SOME CASES OF ACCELERATION OF A MAGNETOHYDRODYNAMIC SHOCK WAVE

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When a shock wave is propagated in a medium with decreasing density, the velocity of the wave front can increase [1]. Below, we shall consider a similar problem for a plasma in a magnetic field.

It has been shown that a strong magnetohydrodynamic shock wave can be accelerated when it moves in an ideal plasma with variable density and variable magnetic-field strength, and the specific energy of a small mass can become very great.

The shock wave is accelerated in proportion to the increase in the Alfvén velocity in front of the wave, so that the Mach number for the shock-wave front will remain constant.

This acceleration mechanism possibly plays a role in the generation of high-energy particles in the plasma around stars and in outer space.

1. **Statement of the problem.** Plane and cylindrical waves are considered. In the initial state, the plasma density ρ_0 and magnetic-field strength H_0 in both cases are distributed as

$$\rho_0(x) = \rho_{00} x^q \qquad H_0(x) = H_{00} x^s \tag{1.1}$$

where ρ_{00}, H_{00}, q, and s are positive constants and x is the distance from the coordinate origin (i.e., the radius in cylindrical coordinates). The magnetic vector is perpendicular to the flow, and in the cylindrical case it is also parallel to the axis. The shock wave (plane or cylindrical) moves toward the coordinate origin: from $x = \infty$ to $x = 0$ (figure). The figure also shows the instantaneous position of the front R and the density jump behind the wave front. The jump at the wave front is considered abrupt, i.e., the structure of the front is not considered.

The time reading is taken assuming that the wave front arrives at point $x = 0$ when $t = 0$. Thus, $0 > t > -\infty$ and $\infty > x \geq 0$.

The viscosity and thermal conductivity of the plasma are assumed to be zero, its electrical conductivity is infinite, and diffusion processes are negligible. In the unperturbed plasma, the magnetic pressure is considered to be much greater than the gas kinetic pressure. The shock wave is assumed to be strong.

2. **Finding self-similar solutions.** The formulated problem admits self-similar solutions. The plasma motion behind the shock-wave front in the one-dimensional nonstationary case is described by the following system of equations [2]:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} + (\nu - 1)\frac{u\rho}{x} = 0;$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} + \frac{H}{4\pi}\frac{\partial H}{\partial x} = 0;$$

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + H\frac{\partial u}{\partial x} + (\nu - 1)\frac{uH}{x} = 0;$$

$$\rho\frac{\partial p}{\partial t} - \gamma p\frac{\partial \rho}{\partial t} + \rho u\frac{\partial p}{\partial x} - \gamma p u\frac{\partial \rho}{\partial x} = 0 \tag{2.1}$$

where γ is the adiabatic exponent, $\nu = 1, 2$, respectively, for the plane and cylindrical cases, and the remaining symbols are conventional.

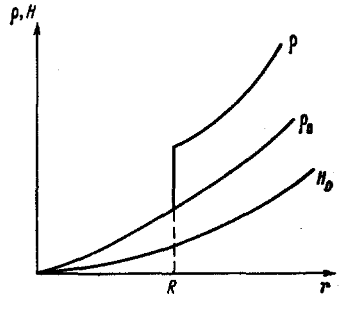
To obtain a self-similar solution, we move from the independent variables (x, t) to (ξ, t), where ξ is a dimensionless variable:

$$\xi = Ax(-t)^\alpha = \frac{x}{R}. \tag{2.2}$$

Here, A and α are constants and R is the coordinate of the shock-wave front.

In relation (2.2), ξ varies within the limits

$$1 \leqslant \xi < \infty.$$



The velocity of the shock-wave front

$$D = \frac{dR}{dt} = A^{-1}\alpha(-t)^{-\alpha-1}. \tag{2.3}$$

The unknown function u, ρ, p, and H of the independent variables (x, t) are found as products of the functions of t and the functions of the self-similar variables ξ:

$$u(x,t) = u_1(t)\varphi(\xi); \qquad \rho(x,t) = \rho_1(t)\psi(\xi);$$
$$p(x,t) = p_1(t)\pi(\xi); \qquad H(x,t) = H_1(t)\chi(\xi). \tag{2.4}$$

Following [1], functions of t are referred to as scales, and functions of ξ are known as representatives.

If we relate the constants to the scales, we can assume that the representatives are dimensionless and satisfy at the wave front (i.e., at $\xi = 1$) the conditions

$$\varphi(1) = \psi(1) = \pi(1) = \chi(1) = 1. \tag{2.5}$$

The scales are easily found, using (2.2) and (2.3), from the known values $u \equiv 0$, ρ_0, $p \equiv 0$, and H_0 immediately ahead of the wave front and the universal conditions at the front of a strong shock wave (the constants A and α still remain undefined)

$$u_1(t) = \frac{2}{\gamma+1}D(t) = \frac{2}{\gamma+1}\frac{\alpha}{A}(-t)^{-\alpha-1};$$

$$\rho_1(t) = \frac{\gamma+1}{\gamma-1}\rho_{00}R^q(t) = \frac{\gamma+1}{\gamma-1}\rho_{00}\frac{1}{A^q}(-t)^{-\alpha q};$$

$$p_1(t) = \frac{2}{\gamma+1}\rho_{00}R^q(t)D^2(t) = \frac{2}{\gamma+1}\rho_{00}\frac{\alpha^2}{A^{2+q}}(-t)^{-\alpha q-2\alpha-2};$$

$$H_1(t) = \frac{\gamma+1}{\gamma-1}H_{00}R^s(t) = \frac{\gamma+1}{\gamma-1}H_{00}\frac{1}{A^s}(-t)^{-\alpha s}. \tag{2.6}$$

If we substitute (2.4) and (2.6) into (2.1), we obtain a system of equations in t and ξ. The condition of separation of variables allows us to determine the index of self-similarity

$$\alpha = \frac{1}{s - {}^1/_2 q - 1}. \tag{2.7}$$

It is important to note that the index of self-similarity is a function only of s and q, in the combination

$$\sigma = s - {}^1/_2 q - 1. \tag{2.8}$$

The values s and q enter the obtained functions of t and R only in this combination (below, it is assumed that $\sigma \neq 1$). For example, the front velocity

$$D = \frac{1}{(\sigma-1)A}(-t)^{-\alpha-1} = \frac{A^{\sigma-1}}{\sigma-1}R^\sigma. \tag{2.9}$$

It follows from the latter expression that a wave moving toward the coordinate origin $R \to 0$ can be accelerated as well as retarded.

Unlimited wave acceleration occurs when

$$\sigma = s - {}^1/_2 q < 0 \ . \tag{2.10}$$

The fact that the index of self-similarity α is a function of s and q in the combination $\sigma = s - q/2$ allows us to give the condition of acceleration or retardation a simple physical interpretation. The local Alfvén velocity in an unperturbed plasma is determined by the same parameter σ:

$$v_A(x) = \frac{H_0}{\sqrt{4\pi\rho_0}} = \frac{H_{00}x^s}{2\sqrt{\pi\rho_{00}}\, x^{1/2 q}} = \frac{H_{00}}{\sqrt{4\pi\rho_{00}}} x^{\sigma} \ . \tag{2.11}$$

From a comparison of (2.9) and (2.11) it follows that the Mach number of the shock wave remains constant as it moves, i.e.,

$$M = \frac{D(x)}{v_A(x)} = \frac{\sqrt{4\pi}\, A^{(\sigma-1)}}{\sigma - 1} \frac{\rho_{00}^{1/2}}{H_{00}} = \text{const} \ . \tag{2.12}$$

Thus, it turns out that for the case in question the acceleration or retardation of the shock wave is determined only by the dependence of the Alfvén velocity on the coordinate x in the unperturbed state. If the Alfvén velocity increases toward the coordinate origin, the wave is accelerated proportionally.

With acceleration, i.e., when $\sigma < 0$, the mass velocity of the plasma increases without bound when $R \to 0$, within the framework of the assumptions that were made. Such accumulation of energy is accompanied, however, by a decrease in mass to zero (in the plane case, the specific mass per unit area), so that the total energy near the coordinate origin nevertheless tends to zero.

If we use the found α value from (2.7), express the constant A in terms of the conserved Mach number M in (2.12), and indicate differentiation with respect to ξ by a prime, we obtain the following system of ordinary differential equations for the representatives, which are functions of the self-similar variable ξ:

$$2\psi\varphi' + [2\varphi - (\gamma+1)\xi]\,\psi' + (\gamma+1)\, q\psi + 2(\nu-1)\frac{\psi\varphi}{\xi} = 0;$$

$$[2\varphi - (\gamma+1)\xi]\,\psi\varphi' + (\gamma-1)\,\pi' + \frac{(\gamma+1)^3}{2(\gamma-1)M^2}\chi\chi' + (\gamma+1)\left(s - \frac{1}{2}q\right)\psi\varphi = 0;$$

$$2\chi\varphi' + [2\varphi - (\gamma+1)\xi]\,\chi' + (\gamma+1)\, s\chi + 2(\nu-1)\frac{\chi\varphi}{\xi} = 0;$$

$$\gamma[2\varphi - (\gamma+1)\xi]\,\pi\psi' - [2\varphi - (\gamma+1)\xi]\,\psi\pi' - (\gamma+1)(2s - q\gamma)\,\pi\psi = 0 \ . \tag{2.13}$$

This system of equations determines the motion behind the wave front. We can integrate it numerically for specific ν, γ, q, s, and M values, which, together with formulas (2.6), gives a complete solution.

The most interesting results are expressed by formulas (2.6); t can be expressed by (2.2) in terms of R and by formula (2.7) for the index of self-similarity. These formulas determine the conditions of wave acceleration, i.e., the energy cumulation.

The index of self-similarity α can be obtained not only from the condition of separation of variables, as was done above, but also from considerations of dimensionality. Under the conditions of the problem, there are only two determining dimensional constants with independent dimensionalities (ρ_{00} and H_{00}), a fact which makes it possible to determine α. It is important to note that α is the same for the plane and cylindrical cases, so that the dependence of wave-front velocity on distance to the coordinate origin is the same, although the flow behind the front differs (see Eq. (2.1)).

Conservation of the Mach number for the moving shock wave (see (2.12)) occurs in both the plane and cylindrical cases. A similar conclusion about conservation of Mach number follows from [3], in which a self-similar magnetohydrodynamic shock wave from a cylindrical explosion was considered. Essentially different problems were solved in [3] and in the present paper.

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